

Time-Dependent Ginzburg–Landau Equation and Boltzmann Transport Equation for Charge-Density-Wave Conductors

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The time-dependent Ginzburg–Landau equation and the Boltzmann transport equation for charge-density-wave (CDW) conductors are derived from a microscopic one-dimensional model by applying the Keldysh Green’s function approach under a quasiclassical approximation. The effects of an external electric field and impurity pinning of the CDW are fully taken into account without relying on a phenomenological argument. These equations simultaneously describe the spatiotemporal dynamics of both the CDW and quasiparticles; thus, they serve as a starting point to develop a general framework to analyze various nonequilibrium phenomena, such as current conversion between the CDW condensate and quasiparticles, in realistic CDW conductors. It is shown that, in typical situations, the equations correctly describe the nonlinear behavior of electric conductivity in a simpler manner.

1. Introduction

The advances in experimental technology have enabled us to closely study the dynamical properties of charge-density-wave (CDW) conductors.¹ Several unique behaviors of CDW conductors have been observed at nonequilibrium.^{2–13} An interesting example is the current conversion between the CDW condensate and quasiparticles mediated by phase slips, which has been the subject of intensive experimental studies.^{14–18} Although plausible models have been proposed for this problem,^{19–31} a reliable picture of the current conversion process has not been established. To fully understand the nonequilibrium behaviors of CDW conductors, we need to describe the dynamics of both the CDW and quasiparticles including the effects of an external electric field and pinning of the CDW due to impurities. In principle, the Keldysh Green’s function approach provides a concrete theoretical framework that fulfills the requirement mentioned above.^{32–38} However, this is not easy to handle, even under a quasiclassical approximation.

To make the framework more tractable, a practical way is to reduce it to a set comprising the time-dependent Ginzburg–Landau (TDGL) equation for the CDW order parameter and the Boltzmann transport (BT) equation for charge and current densities by expanding it with respect to the CDW order parameter. These equations are written in the form of differential equations, and hence are easy to handle, although their application may be restricted to the regime where the magnitude of the CDW order parameter is sufficiently small. Such a framework based on the TDGL and BT equations has been fully developed and is widely used in the field of supercon-

ductivity,³⁹ while it remains inadequate in the field of CDW conductors. Previously, the TDGL equation for CDW conductors has been proposed and used to analyze dynamical features of the CDW.^{40–42} However, the proposed TDGL equation describes only the CDW degrees of freedom and the role of quasiparticles is neglected. The BT equation for CDW conductors has been introduced by several authors.^{35,43,44} However, its application is restricted to the case without pinning of the CDW.

In this paper, we derive the TDGL and BT equations for CDW conductors starting from a microscopic one-dimensional model. The resulting equations fully describe the dynamics of both the CDW and quasiparticles including the effects of an external electric field and impurity pinning in one-dimensional situations, and serve as a starting point to develop a general framework to analyze nonequilibrium phenomena in realistic CDW conductors. They are derived by applying the Keldysh Green’s function approach^{45,46} under a quasiclassical approximation^{47,48} without relying on a phenomenological argument. In addition to the ordinary assumption that the magnitude of the CDW order parameter $\Delta(x, t)$ is much smaller than the temperature T , we assume that the nonequilibrium distribution of quasiparticles is described by the Fermi-Dirac function with a space- and time-dependent chemical potential $\mu_{\pm}(x, t)$, where $\mu_{+}(x, t)$ [$\mu_{-}(x, t)$] denotes the chemical potential for the right-going (left-going) quasiparticles [see Eq. (32)]. The dynamical variables in our problem are $\Delta(x, t)$, $\mu_{+}(x, t)$, and $\mu_{-}(x, t)$, by which the dynamics of the CDW and quasiparticles are described including the effects of an external electric field and impurity pinning.

In the next section, we present a one-dimensional model for CDW conductors and introduce the Keldysh Green's function with its equation of motion under a quasiclassical approximation. In Sect. 3, we derive the TDGL and BT equations by using the standard procedure of the Keldysh Green's function approach developed in the field of superconductivity. In Sect. 4, we analyze the nonlinear behavior of electric conductivity in typical situations within the framework of the TDGL and BT equations. It is shown that the previously reported result is correctly reproduced in a simpler manner. The last section is devoted to a short summary. Preliminary results of this work have been briefly reported in Refs. 49 and 50. We ignore the spin degree of freedom and set $\hbar = k_B = 1$ throughout this paper.

2. Model and Formulation

To present a one-dimensional model for CDW conductors, it is convenient to decompose the electron field operator $\psi(x)$ into the right-going and left-going components: $\psi(x) = e^{ik_F x} \psi_+(x) + e^{-ik_F x} \psi_-(x)$ with k_F being the Fermi wave number. The model Hamiltonian is given by $H = H_0 + H_{\text{imp}}$ with

$$H_0 = \int dx \left\{ \psi_+^\dagger(x) d_+(x, t) \psi_+(x) + \psi_-^\dagger(x) d_-(x, t) \psi_-(x) + g u(x, t) e^{-iQx} \psi_+^\dagger(x) \psi_-(x) + \text{h.c.} \right\}, \quad (1)$$

$$H_{\text{imp}} = \int dx \left\{ \psi_+^\dagger(x) V_{\text{imp}}(x) \psi_+(x) + \psi_-^\dagger(x) V_{\text{imp}}(x) \psi_-(x) + \psi_+^\dagger(x) e^{-iQx} V_{\text{imp}}(x) \psi_-(x) + \text{h.c.} \right\}, \quad (2)$$

where

$$d_\pm(x, t) = \mp i v_F [\partial_x + i e A(x, t)] + \Phi(x, t), \quad (3)$$

$Q = 2k_F$, g is the coupling constant between electrons and phonons, and $u(x, t)$ is the lattice displacement, which is directly connected with phonon degrees of freedom. Here, $\Phi(x, t) = -e\phi(x, t)$ with ϕ being the scalar potential. An electric field is expressed in terms of Φ and A in a gauge-invariant manner [see Eq. (39)]. We assume that the impurity potential V_{imp} is given by

$$V_{\text{imp}}(x) = \sum_i v(x - x_i), \quad (4)$$

where x_i denotes the location of the i th impurity.

In the previous theoretical treatment based on the Keldysh Green's function approach,^{32–37} the influence of H_{imp} is taken into account as a correction to the self-energy under the averaging over impurity configurations (i.e., disorder average). As a first-order correction to the self-energy vanishes under the disorder average, a nonvanishing contribution originates from a second-order correction, which describes quasiparticle scattering. An apparent drawback of this treatment is that impurity pin-

ning completely disappears in the resulting framework. This is simply because the translational invariance of the system is restored under the disorder average in spite of the fact that a spatial inhomogeneity is indispensable for the pinning of CDWs.

The pinning of CDWs is induced by backward scattering between right-going and left-going electrons,^{51–54} thus, the corresponding contribution should arise from the third and fourth terms in H_{imp} containing

$$\zeta(x) \equiv e^{-iQx} V_{\text{imp}}(x) \quad (5)$$

or its complex conjugate. We capture the effect of impurity pinning by explicitly incorporating it into our consideration. Note that $\zeta(x)$ vanishes under the disorder average; thus, we keep in mind that it is defined under a given impurity configuration. Since $\zeta(x)$ plays a similar role to $g u(x, t) e^{-iQx}$ in H_0 , it is natural to define the order parameter $\Delta(x, t)$ for the CDW by adding $\zeta(x)$ to $g \langle u(x, t) \rangle e^{-iQx}$ as

$$\Delta(x, t) \equiv g \langle u(x, t) \rangle e^{-iQx} + \zeta(x), \quad (6)$$

where $\langle \cdots \rangle$ denotes the thermal average.⁵⁵ Note that $\zeta(x)$ in the definition of $\Delta(x, t)$ can be regarded as a first-order correction to the self-energy. We hereafter consider the following effective Hamiltonian:

$$H_{\text{eff}} = \int dx \left\{ \psi_+^\dagger(x) d_+(x, t) \psi_+(x) + \psi_-^\dagger(x) d_-(x, t) \psi_-(x) + \Delta(x, t) \psi_+^\dagger(x) \psi_-(x) + \text{h.c.} \right\}, \quad (7)$$

instead of H_0 . Although $\zeta(x)$ seemingly disappears in H_{eff} , it certainly describes the impurity pinning via the self-consistency equation for $\Delta(x, t)$ [see Eq. (19)]. In the argument given below, the effect of quasiparticle scattering is taken into account as an ordinary second-order correction to the self-energy [see Eq. (17)].

To describe the dynamics of the CDW and quasiparticles governed by H_{eff} with H_{imp} , we introduce the Keldysh Green's function consisting of the Keldysh, retarded, and advanced components:⁴⁵

$$G_{\alpha\beta}^K(x, t; x', t') = -i \times \left\langle \psi_\alpha(x, t) \psi_\beta^\dagger(x', t') - \psi_\beta^\dagger(x', t') \psi_\alpha(x, t) \right\rangle, \quad (8)$$

$$G_{\alpha\beta}^R(x, t; x', t') = -i \Theta(t - t') \times \left\langle \psi_\alpha(x, t) \psi_\beta^\dagger(x', t') + \psi_\beta^\dagger(x', t') \psi_\alpha(x, t) \right\rangle, \quad (9)$$

$$G_{\alpha\beta}^A(x, t; x', t') = +i \Theta(t' - t) \times \left\langle \psi_\alpha(x, t) \psi_\beta^\dagger(x', t') + \psi_\beta^\dagger(x', t') \psi_\alpha(x, t) \right\rangle, \quad (10)$$

where $\alpha, \beta (= \pm)$ specify the right-going and left-going components, and $\Theta(t)$ is Heaviside's step function. In accordance with the effective Hamiltonian, the matrix Green's functions $\hat{G}^X(x, t; x', t')$ ($X = K, R, A$), defined

by

$$\hat{G}^X(x, t; x', t') = \begin{bmatrix} G_{++}^X & G_{+-}^X \\ G_{-+}^X & G_{--}^X \end{bmatrix}, \quad (11)$$

satisfy

$$\begin{aligned} \hat{D}(x, t) \hat{G}^K(x, t; x', t') &= \left\{ \hat{\Sigma}_*^R \otimes \hat{G}^K \right\} (x, t; x', t') \\ &+ \left\{ \hat{\Sigma}_*^K \otimes \hat{G}^A \right\} (x, t; x', t'), \end{aligned} \quad (12)$$

$$\begin{aligned} \hat{D}(x, t) \hat{G}^{R,A}(x, t; x', t') &= \hat{\sigma}_0 \delta(x - x') \delta(t - t') \\ &+ \left\{ \hat{\Sigma}_*^{R,A} \otimes \hat{G}^{R,A} \right\} (x, t; x', t'), \end{aligned} \quad (13)$$

where

$$\begin{aligned} \hat{D}(x, t) &= \hat{\sigma}_0 [i\partial_t - \Phi(x, t)] + i\hat{\sigma}_z v_F [\partial_x + ieA(x, t)] \\ &- \hat{\Delta}_*(x, t) \end{aligned} \quad (14)$$

with

$$\hat{\Delta}_*(x, t) = \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix}, \quad (15)$$

and $\{A \otimes B\}(x, t; x', t')$ denotes

$$\begin{aligned} \{A \otimes B\}(x, t; x', t') &= \int dx_1 \int dt_1 A(x, t; x_1, t_1) B(x_1, t_1; x', t'). \end{aligned} \quad (16)$$

Here and hereafter, we use $\hat{\sigma}_0$ and $\hat{\sigma}_i$ ($i = x, y, z$) to denote the 2×2 unit matrix and the i -component of the Pauli matrix, respectively. The self-energy $\hat{\Sigma}_*^X$ describing the impurity scattering of quasiparticles is given by

$$\hat{\Sigma}_*^X(x, t; x', t') = \left\langle \hat{V}_{\text{imp}}(x) \hat{G}^X(x, t; x', t') \hat{V}_{\text{imp}}(x') \right\rangle_{\text{imp}}, \quad (17)$$

where $\langle \cdots \rangle_{\text{imp}}$ denotes the disorder average and

$$\hat{V}_{\text{imp}}(x) = V_{\text{imp}}(x) \begin{bmatrix} 1 & e^{-iQx} \\ e^{iQx} & 1 \end{bmatrix}. \quad (18)$$

The equations for the matrix Green's functions are supplemented by the self-consistency equation for the order parameter:

$$\Delta(x, t) = 4\pi i v_F \lambda G_{+-}^K(x, t; x', t')|_{x' \rightarrow x, t' \rightarrow t} + \zeta(x), \quad (19)$$

where λ is a dimensionless coupling constant. The derivation of Eq. (19) is outlined in Appendix A. We see below that $\zeta(x)$ in Eq. (19) gives rise to impurity pinning.

With the Fourier transform of \hat{G}^X ,

$$\begin{aligned} \hat{G}^X(x, k; t, \epsilon) &\equiv \int dy e^{-iky} \int d\tau e^{i\epsilon\tau} \\ &\times \hat{G}^X\left(x + \frac{y}{2}, t + \frac{\tau}{2}; x - \frac{y}{2}, t - \frac{\tau}{2}\right), \end{aligned} \quad (20)$$

we introduce the quasiclassical Green's function:⁴⁷

$$\hat{g}^X(x; t, \epsilon) = \hat{\sigma}_z \frac{iv_F}{\pi} \int dk \hat{G}^X(x, k; t, \epsilon), \quad (21)$$

where a diverging contribution must be subtracted. The self-consistency equation is rewritten as

$$\Delta(x, t) = \lambda \int d\epsilon g_{+-}^K(x; t, \epsilon) + \zeta(x). \quad (22)$$

The charge density $\rho(x, t)$ and the current density $j(x, t)$ are expressed as

$$\rho(x, t) = \frac{e}{\pi v_F} \left(\frac{1}{8} \int d\epsilon \text{tr} \{ \hat{\sigma}_z \hat{g}^K(x; t, \epsilon) \} + \Phi(x, t) \right), \quad (23)$$

$$j(x, t) = \frac{e}{8\pi} \int d\epsilon \text{tr} \{ \hat{g}^K(x; t, \epsilon) \}. \quad (24)$$

Within a quasiclassical approximation, we can derive a kinetic equation for $\hat{g}^X(x; t, \epsilon)$ from Eqs. (12) and (13):

$$\begin{aligned} (\epsilon - \Phi) [\hat{\sigma}_z, \hat{g}^X]_- + \frac{i}{2} [\hat{\sigma}_z, \dot{\hat{g}}^X]_+ + iv_F \partial_x \hat{g}^X + [\hat{\Delta}, \hat{g}^X]_- \\ + ie v_F \dot{\hat{\Delta}} \partial_\epsilon \hat{g}^X + \frac{i}{2} \dot{\hat{\Phi}} [\hat{\sigma}_z, \partial_\epsilon \hat{g}^X]_+ - \frac{i}{2} [\dot{\hat{\Delta}}, \partial_\epsilon \hat{g}^X]_+ \\ - \frac{1}{8} [\ddot{\hat{\Delta}}, \partial_\epsilon^2 \hat{g}^X]_- - [\hat{\Sigma}^X, \hat{g}^X]_- = 0, \end{aligned} \quad (25)$$

where $\hat{\Delta} \equiv -\hat{\Delta}_* \hat{\sigma}_z$. The self-energy $\hat{\Sigma}^X \equiv \hat{\Sigma}_*^X \hat{\sigma}_z$ is expressed as³²

$$\hat{\Sigma}^X(x; t, \epsilon) = -\frac{i}{2} \left[\nu_1 \hat{\sigma}_z \hat{g}^X \hat{\sigma}_z - \frac{\nu_2}{2} (\hat{\sigma}_x \hat{g}^X \hat{\sigma}_x + \hat{\sigma}_y \hat{g}^X \hat{\sigma}_y) \right], \quad (26)$$

where ν_1 and ν_2 respectively characterize the strength of forward and backward scattering. They are given by $\nu_1 = n_{\text{imp}} |v(0)|^2 / v_F$ and $\nu_2 = n_{\text{imp}} |v(Q)|^2 / v_F$, where n_{imp} and $v(q)$ are respectively the density of impurities and the Fourier transform of the impurity potential $v(x)$. For later convenience, we define the elastic relaxation time τ and the transport relaxation time τ_{tr} as

$$\frac{1}{2\tau} = \nu_1 + \frac{\nu_2}{2}, \quad (27)$$

$$\frac{1}{2\tau_{\text{tr}}} = \nu_2. \quad (28)$$

The self-energy term in Eq. (25) for the Keldysh component reads

$$[\hat{\Sigma}^K, \hat{g}^K]_- = \hat{\Sigma}^R \hat{g}^K + \hat{\Sigma}^K \hat{g}^A - \hat{g}^R \hat{\Sigma}^K - \hat{g}^K \hat{\Sigma}^A. \quad (29)$$

We adopt the following approximate expression for $\hat{g}^{K,46,48}$

$$\hat{g}^K(x; t, \epsilon) = \hat{g}^R \hat{n} - \hat{n} \hat{g}^A - \frac{i}{2} [\partial_t \hat{g}^R \partial_\epsilon \hat{n} + \partial_\epsilon \hat{n} \partial_t \hat{g}^A]$$

$$+ \frac{i}{2} [\partial_\epsilon \hat{g}^R \partial_t \hat{n} + \partial_t \hat{n} \partial_\epsilon \hat{g}^A] - \frac{1}{8} [\partial_t^2 \hat{g}^R \partial_\epsilon^2 \hat{n} - \partial_\epsilon^2 \hat{n} \partial_t^2 \hat{g}^A] \quad (30)$$

with

$$\hat{n}(x; t, \epsilon) = \begin{bmatrix} 1 - 2f_+(x; t, \epsilon) & 0 \\ 0 & 1 - 2f_-(x; t, \epsilon) \end{bmatrix}, \quad (31)$$

where f_+ (f_-) denotes the distribution function for right-going (left-going) quasiparticles. Now we employ the assumption that f_+ and f_- are expressed as

$$f_\pm(x, t, \epsilon) = f_{\text{FD}}(\epsilon - \Phi(x, t) - \mu_\pm(x, t)), \quad (32)$$

where $f_{\text{FD}}(\epsilon)$ is the Fermi-Dirac function and μ_+ (μ_-) is the chemical potential for right-going (left-going) quasiparticles.

3. Derivation of the TDGL and BT Equations

To derive the TDGL and BT equations, we solve Eq. (25) for \hat{g}^R and \hat{g}^A . With the explicit representation of

$$\hat{g}^R(x; t, \epsilon) = \begin{bmatrix} g^R(x; t, \epsilon) & f^R(x; t, \epsilon) \\ \bar{f}^R(x; t, \epsilon) & \bar{g}^R(x; t, \epsilon) \end{bmatrix}, \quad (33)$$

Eq. (25) for \hat{g}^R is decomposed into the four equations given in Appendix B. Assuming that $|\Delta| \ll T$, we approximately solve them with the help of the normalization condition in the static limit:

$$(g^X)^2 + f^X \bar{f}^X = (\bar{g}^X)^2 + f^X \bar{f}^X = 1, \quad (34)$$

where $X = R$ or A . We find that the matrix elements are given by

$$\begin{aligned} g^R(x; t, \epsilon) &= 1 + \frac{|\Delta|^2}{2(\epsilon - \Phi + \frac{i}{2\tau})^2} \\ &\quad - \frac{iv_F}{4(\epsilon - \Phi + \frac{i}{2\tau})^3} (\Delta^* \partial_x \Delta - \Delta \partial_x \Delta^*) \\ &\quad + \frac{i}{4(\epsilon - \Phi + \frac{i}{2\tau})^3} (\Delta^* \partial_t \Delta - \Delta \partial_t \Delta^*), \end{aligned} \quad (35)$$

$$\begin{aligned} \bar{g}^R(x; t, \epsilon) &= -1 - \frac{|\Delta|^2}{2(\epsilon - \Phi + \frac{i}{2\tau})^2} \\ &\quad + \frac{iv_F}{4(\epsilon - \Phi + \frac{i}{2\tau})^3} (\Delta^* \partial_x \Delta - \Delta \partial_x \Delta^*) \\ &\quad + \frac{i}{4(\epsilon - \Phi + \frac{i}{2\tau})^3} (\Delta^* \partial_t \Delta - \Delta \partial_t \Delta^*), \end{aligned} \quad (36)$$

$$\begin{aligned} f^R(x; t, \epsilon) &= \frac{\Delta}{\epsilon - \Phi + \frac{i}{2\tau}} - \frac{iv_F}{2(\epsilon - \Phi + \frac{i}{2\tau})^2} \partial_x \Delta \\ &\quad - \frac{\epsilon - \Phi}{2(\epsilon - \Phi + \frac{i}{2\tau})^4} \Delta |\Delta|^2 \\ &\quad - \frac{iv_F e E}{2(\epsilon - \Phi + \frac{i}{2\tau})^3} \Delta - \frac{v_F^2}{4(\epsilon - \Phi + \frac{i}{2\tau})^3} \partial_x^2 \Delta, \end{aligned} \quad (37)$$

$$\begin{aligned} \bar{f}^R(x; t, \epsilon) &= -\frac{\Delta^*}{\epsilon - \Phi + \frac{i}{2\tau}} - \frac{iv_F}{2(\epsilon - \Phi + \frac{i}{2\tau})^2} \partial_x \Delta^* \\ &\quad - \frac{\epsilon - \Phi}{2(\epsilon - \Phi + \frac{i}{2\tau})^4} \Delta^* |\Delta|^2 \\ &\quad - \frac{iv_F e E}{2(\epsilon - \Phi + \frac{i}{2\tau})^3} \Delta^* + \frac{v_F^2}{4(\epsilon - \Phi + \frac{i}{2\tau})^3} \partial_x^2 \Delta^*, \end{aligned} \quad (38)$$

where E denotes the electric field defined by

$$E(x, t) = \frac{1}{e} \partial_x \Phi(x, t) - \partial_t A(x, t). \quad (39)$$

The advanced function \hat{g}^A is obtained via the relation

$$\hat{g}^A(x; t, \epsilon) = -\hat{g}^R(x; t, \epsilon)|_{\frac{i}{2\tau} \rightarrow -\frac{i}{2\tau}}. \quad (40)$$

Approximating \hat{g}^K by the first term of the gradient expansion of Eq. (30), namely, $\hat{g}^K = \hat{g}^R \hat{n} - \hat{n} \hat{g}^A$, and substituting this with Eqs. (35)–(38) into Eqs. (23) and (24), we readily find that the charge and current densities are given by

$$\begin{aligned} \rho(x, t) &= -\frac{e}{2\pi v_F} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right) (\mu_+ + \mu_-) \\ &\quad + ie \frac{7\zeta(3)}{16\pi^3 T^2} (\Delta^* \partial_x \Delta - \Delta \partial_x \Delta^*), \end{aligned} \quad (41)$$

$$\begin{aligned} j(x, t) &= -\frac{e}{2\pi} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right) (\mu_+ - \mu_-) \\ &\quad - ie \frac{7\zeta(3)}{16\pi^3 T^2} (\Delta^* \partial_t \Delta - \Delta \partial_t \Delta^*). \end{aligned} \quad (42)$$

We substitute Eq. (30) with Eqs. (35)–(38) into the self-consistency equation [Eq. (22)]. After tedious but straightforward calculations with the assumption of $\tau T \gg 1$, we find that the TDGL equation is expressed as

$$\begin{aligned} &\left(1 - \frac{7\zeta(3)}{\pi^3 \tau T} \right) [\partial_t \Delta + i(\mu_+ - \mu_-) \Delta] \\ &= \frac{8T}{\pi} \left(1 - \frac{\pi}{8\tau T} \right) \left(1 - \frac{T}{T_c} \right) \Delta + \frac{2T}{\pi \lambda} \zeta(x) \\ &\quad + \frac{7\zeta(3)}{2\pi^3 T} \left\{ v_F \partial_x - i(\mu_+ + \mu_-) \right\}^2 \Delta \\ &\quad + [\partial_t + i(\mu_+ - \mu_-)]^2 \Delta - 2\Delta |\Delta|^2 \Big\} \\ &\quad + i \frac{7\zeta(3)}{2\pi^3 T} \left\{ v_F \partial_x (\mu_+ + \mu_-) + \partial_t (\mu_+ - \mu_-) + 2v_F e E \right\} \Delta. \end{aligned} \quad (43)$$

The second term in the right-hand side represents the impurity pinning of the CDW.

Let us return to the kinetic equation for \hat{g}^K [Eq. (25)]. Taking the trace of this equation with the help of Eqs. (26) and (29) and then integrating it over ϵ , we

obtain the following equation:

$$\int d\epsilon \operatorname{tr} \left\{ \hat{\sigma}_z \partial_t \hat{g}^K + v_F \partial_x \hat{g}^K + \left(v_F e \dot{A} - \dot{\hat{\Delta}} + \hat{\sigma}_z \dot{\Phi} \right) \partial_\epsilon \hat{g}^K \right\} = 0. \quad (44)$$

Using Eqs. (23) and (24) with $\int d\epsilon \partial_\epsilon \hat{g}^K = 4\hat{\sigma}_z$, we can show that Eq. (44) is equivalent to the continuity equation³⁵

$$\partial_t \rho + \partial_x j = 0. \quad (45)$$

In a similar manner, we take the trace of the kinetic equation multiplied by $\hat{\sigma}_z$ with the help of Eqs. (26) and (29) and then integrate it over ϵ . This leads to

$$\begin{aligned} \int d\epsilon \operatorname{tr} \left\{ \partial_t \hat{g}^K + v_F \hat{\sigma}_z \partial_x \hat{g}^K - i \left(\hat{\sigma}_z \hat{\Delta} - \hat{\Delta} \hat{\sigma}_z \right) \hat{g}^K \right. \\ \left. + \left(v_F e \dot{A} \hat{\sigma}_z + \dot{\Phi} \right) \partial_\epsilon \hat{g}^K + \frac{i}{8} \left(\hat{\sigma}_z \ddot{\hat{\Delta}} - \ddot{\hat{\Delta}} \hat{\sigma}_z \right) \partial_\epsilon^2 \hat{g}^K \right. \\ \left. - \frac{i}{2} \nu_2 \left(\hat{g}^R - \hat{g}^A \right) \left(\hat{\sigma}_x \hat{g}^K \hat{\sigma}_y - \hat{\sigma}_y \hat{g}^K \hat{\sigma}_x \right) \right\} = 0. \quad (46) \end{aligned}$$

Using Eqs. (23) and (24) with $\int d\epsilon \partial_\epsilon \hat{g}^K = 4\hat{\sigma}_z$ and $\int d\epsilon \partial_\epsilon^2 \hat{g}^K = 0$, we can show that the above equation gives rise to the BT equation:

$$\begin{aligned} \frac{\pi}{e} \left(\partial_t j + v_F^2 \partial_x \rho \right) - v_F e E \\ = \frac{1}{2\tau_{\text{tr}}} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right)^2 (\mu_+ - \mu_-) + \eta, \quad (47) \end{aligned}$$

where τ_{tr} is defined in Eq. (28) and

$$\eta = \frac{i}{8} \int d\epsilon \operatorname{tr} \left\{ \left(\hat{\sigma}_z \hat{\Delta} - \hat{\Delta} \hat{\sigma}_z \right) \hat{g}^K \right\}. \quad (48)$$

The first term in the right-hand side of Eq. (47) describes the relaxation of quasiparticles due to impurity scattering. The expression for η is determined by using Eq. (22) as

$$\begin{aligned} \eta = \frac{i}{4\lambda} (\zeta^* \Delta - \zeta \Delta^*) \\ = -\frac{1}{2\lambda} V_{\text{imp}}(x) |\Delta(x, t)| \sin(Qx + \theta(x, t)) \quad (49) \end{aligned}$$

with

$$\theta(x, t) = \arg\{\Delta(x, t)\}. \quad (50)$$

This clearly indicates that η describes the relaxation of the CDW motion due to impurity pinning.

In the remainder of this section, we briefly consider the effective pinning potential giving rise to the pinning term in the TDGL equation. As the TDGL equation should be expressed in the form of

$$\frac{\pi}{8v_F T} \left(1 - \frac{7\zeta(3)}{\pi^3 \tau T} \right) \partial_t \Delta = -\frac{\delta F(\Delta, \Delta^*)}{\delta \Delta^*} \quad (51)$$

in terms of the free energy, written as

$$\begin{aligned} F = \int dx \frac{1}{v_F} \left[-\left(1 - \frac{\pi}{8\tau T} \right) \left(1 - \frac{T}{T_c} \right) \Delta \Delta^* \right. \\ \left. + \frac{7\zeta(3)}{16\pi^2 T^2} (\Delta \Delta^*)^2 + \dots \right] + F_{\text{pin}}, \quad (52) \end{aligned}$$

the pinning potential F_{pin} is identified as

$$\begin{aligned} F_{\text{pin}} = -\frac{1}{4\lambda v_F} \int dx (\zeta^* \Delta + \zeta \Delta^*) \\ = -\frac{1}{2\lambda v_F} \int dx V_{\text{imp}}(x) |\Delta(x)| \cos(Qx + \theta(x)). \quad (53) \end{aligned}$$

The λ dependence of F_{pin} is consistent with Eq. (6.9) of Ref. 53. With the expression for V_{imp} [Eq. (4)], the pinning potential is rewritten in the following well-known form:^{51,52}

$$F_{\text{pin}} = -\frac{1}{2\lambda v_F} \int dx \sum_i v(x - x_i) |\Delta(x)| \cos(Qx + \theta(x)). \quad (54)$$

4. Application of the TDGL and BT Equations

The TDGL equation [Eq. (43)] with Eq. (5) and the BT equation [Eq. (47)] with Eq. (49) are the central result of this paper. Although the effect of the Coulomb interaction is not explicitly considered in their derivation, we can take account of it including the screening effect due to quasiparticles by supplementing them by Gauss's law.^{49,50} The most interesting application of the TDGL and BT equations is to use them for numerical simulations of the nonequilibrium dynamics of the CDW and quasiparticles in phase slip processes.⁵⁰ Such a numerical study will be reported in a forthcoming publication. Here, by using the two equations, we analyze the dc electric conductivity of CDW conductors in the situation where a space- and time-independent current density (i.e., $j = \text{constant}$) is supplied to the system. The factor $(\tau T)^{-1}$ in the TDGL equation is ignored in the following argument as it does not play an important role.

Firstly, we consider the case where the electric field E is much smaller than the threshold value E_{th} for CDW depinning and hence the CDW is completely pinned. That is, $\theta(x, t)$ as well as $|\Delta(x, t)|$ is independent of t but varies as a function of x according to the pinning potential. As only quasiparticles contribute to the current density in this case, the chemical potentials for right-going and left-going quasiparticles should satisfy $\mu_+ - \mu_- = \text{constant}$. The TDGL equation reads

$$\begin{aligned} i(\mu_+ - \mu_-) |\Delta| \\ = \frac{8T}{\pi} \left(1 - \frac{T}{T_c} \right) |\Delta| + \frac{2T}{\pi\lambda} |\zeta| e^{-i(Qx + \theta)} \\ + \frac{7\zeta(3)}{2\pi^3 T} \left[v_F^2 (\partial_x^2 |\Delta| - (\partial_x \theta)^2 |\Delta|) \right] \end{aligned}$$

$$\begin{aligned}
& + 2v_F(\mu_+ + \mu_-)(\partial_x \theta)|\Delta| - 2(\mu_+^2 + \mu_-^2)|\Delta| - 2|\Delta|^3 \Big] \\
& + i \frac{7\zeta(3)}{2\pi^3 T} \left[v_F^2 ((\partial_x^2 \theta)|\Delta| + 2(\partial_x \theta)(\partial_x |\Delta|)) \right. \\
& \quad \left. - 2v_F(\mu_+ + \mu_-)\partial_x |\Delta| + 2v_F e E |\Delta| \right] \quad (55)
\end{aligned}$$

and the BT equation is expressed as

$$\begin{aligned}
& - \frac{v_F}{2} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right) \partial_x (\mu_+ + \mu_-) \\
& - \frac{7\zeta(3)}{4\pi^2 T^2} v_F \left[\frac{v_F}{2} (\partial_x^2 \theta)|\Delta|^2 + v_F (\partial_x \theta)(\partial_x |\Delta|)|\Delta| \right. \\
& \quad \left. - (\mu_+ + \mu_-)(\partial_x |\Delta|)|\Delta| \right] - v_F e E \\
& = \frac{1}{2\tau_{tr}} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right)^2 (\mu_+ - \mu_-) \\
& - \frac{1}{2\lambda} |\zeta| |\Delta| \sin(Qx + \theta). \quad (56)
\end{aligned}$$

We take $\partial_x (\mu_+ + \mu_-)$ into consideration in Eq. (56) since the spatial variation of $\theta(x, t)$ leads to an inhomogeneity in the charge density of the CDW, which then induces a spatial variation of the quasiparticle charge density determined by $\mu_+ + \mu_-$. This phenomenon (i.e., the screening of the charge density due to quasiparticles) is described by Gauss's law. However, its details including the resulting profile of Δ do not affect the argument given below. The imaginary part of Eq. (55) yields

$$\begin{aligned}
& (\mu_+ - \mu_-)|\Delta| \\
& = \frac{7\zeta(3)}{4\pi^3 T} v_F \left[\frac{v_F}{2} ((\partial_x^2 \theta)|\Delta| + (\partial_x \theta)(\partial_x |\Delta|)) \right. \\
& \quad \left. - (\mu_+ + \mu_-)\partial_x |\Delta| + eE|\Delta| \right] - \frac{2T}{\pi\lambda} |\zeta| \sin(Qx + \theta). \quad (57)
\end{aligned}$$

The combination of the last two equations yields

$$\begin{aligned}
& -v_F e \tilde{E} = \frac{1}{2\tau_{tr}} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right) \left(1 + \frac{\pi\tau_{tr}|\Delta|^2}{2T} \right) \\
& \quad \times (\mu_+ - \mu_-), \quad (58)
\end{aligned}$$

where

$$\tilde{E} = E + \frac{1}{2e} \partial_x (\mu_+ + \mu_-). \quad (59)$$

Note that \tilde{E} corresponds to the derivative of the electrochemical potential. By using this relation, we eliminate $\mu_+ - \mu_-$ in the expression for the current density, which reads

$$j = -\frac{e}{2\pi} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right) (\mu_+ - \mu_-) \quad (60)$$

in this case. We finally find that $j = \sigma \tilde{E}$ with the con-

ductivity σ given by

$$\sigma = \sigma_N \left(1 - \frac{\pi\tau_{tr}|\Delta|^2}{2T} \right), \quad (61)$$

where $\sigma_N \equiv e^2 v_F \tau_{tr} / \pi$ is the conductivity in the normal state. We observe that the conductivity is smaller than the normal-state value. This should not simply be attributed to the suppression of the density of states, which certainly decreases j as shown in Eq. (60) but effectively increases τ_{tr} according to Eq. (56). As the factor of the increase in τ_{tr} is larger than the factor of the decrease in j , the conductivity is enhanced, contrary to the above result, if only these two changes are taken into consideration. The reduction of σ below the normal-state value is mainly caused by the screening of the electric field due to the CDW, which dominates the effect of the suppression of the density of states.

Secondly, we consider the case where $E \gg E_{th}$ and hence the CDW moves in the direction of acceleration. In this case, the CDW motion can be approximated as uniform sliding as long as our attention is focused on the dc conductivity. That is, $|\Delta(x, t)| = \text{constant}$ and the phase increases with the angular velocity ω as $\theta(x, t) = -\omega t + \text{constant}$. In accordance with this approximation, we assume that $\mu_+ + \mu_- = 0$, resulting in $\rho = 0$. Note that the current density arises from both the CDW and quasiparticles in this case. The TDGL equation reads

$$\begin{aligned}
& i(-\omega + \mu_+ - \mu_-)|\Delta| \\
& = \frac{8T}{\pi} \left(1 - \frac{T}{T_c} \right) |\Delta| + \frac{7\zeta(3)}{2\pi^3 T} \left[-(-\omega + \mu_+ - \mu_-)^2 \right. \\
& \quad \left. - 2|\Delta|^2 + i2v_F e E \right] |\Delta|, \quad (62)
\end{aligned}$$

and the BT equation is expressed as

$$-v_F e E = \frac{1}{2\tau_{tr}} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right)^2 (\mu_+ - \mu_-), \quad (63)$$

where the pinning term, which oscillates with the frequency ω , is ignored in each equation since our attention is focused on the dc conductivity. The imaginary part of Eq. (62) yields

$$-\omega + \mu_+ - \mu_- = \frac{7\zeta(3)}{\pi^3 T} v_F e E. \quad (64)$$

Now, we derive the dc conductivity from the expression for the current density:

$$j = -\frac{e}{2\pi} \left(1 - \frac{7\zeta(3)}{4\pi^2 T^2} |\Delta|^2 \right) (\mu_+ - \mu_-) - e \frac{7\zeta(3)}{8\pi^3 T^2} \omega |\Delta|^2, \quad (65)$$

where the second term corresponds to the CDW current. Using Eqs. (63) and (64), we can rewrite $\mu_+ - \mu_-$ and ω in Eq. (65) in terms of E , and we find that $j = \sigma E$ with

$$\sigma = \sigma_N \left[1 + \frac{7\zeta(3)|\Delta|^2}{2\pi^2 T^2} \left(1 + \frac{7\zeta(3)}{4\pi^3 \tau_{tr} T} \right) \right]. \quad (66)$$

The conductivity is slightly larger than the normal-state value.⁵⁶ This is mainly caused by the increase in τ_{tr} due to the suppression of the density of states without the decrease in j , which is compensated by the additional contribution from the sliding CDW.

We see that σ is smaller than σ_N in the regime of $E \ll E_{th}$ while it is slightly larger than σ_N in the opposite regime of $E \gg E_{th}$. This clearly indicates the nonlinear behavior of electric conductivity in CDW conductors. It should be mentioned that Eq. (61) is equivalent to Eq. (34) of Ref. 32 and that Eq. (66) is also equivalent to the equation presented just below Eq. (35) of Ref. 32. Our derivation is much simpler than that in Ref. 32, which was based on the Keldysh Green's function approach under a quasiclassical approximation. Furthermore, the effect of impurity pinning was treated in a phenomenological manner in Ref. 32, while we explicitly take it into consideration. In these respects, our theoretical framework has an advantage over the previous one.

5. Summary

We have derived the time-dependent Ginzburg–Landau equation and the Boltzmann transport equation for charge-density-wave (CDW) conductors from a microscopic one-dimensional model by applying the Keldysh Green's function approach under a quasiclassical approximation. We have succeeded in introducing the pinning term without relying on a phenomenological argument. These equations simultaneously describe the spatiotemporal dynamics of both the CDW and quasiparticles; thus, they can be widely used to analyze various nonequilibrium phenomena that are associated with both their degrees of freedom. For example, they enable us to numerically simulate the dynamics of the CDW and quasiparticles in phase slip processes including the effects of an external electric field and impurity pinning.⁵⁰ An extension of this framework to two- and three-dimensional cases is the most important achievement to be accomplished.

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Appendix A: Derivation of the Self-Consistency Equation

In this appendix, we derive the self-consistency equation for $\Delta(x, t)$ following the argument in Ref. 32. Let us consider the Hamiltonian $H' = H_{ph} + H_{e-p}$, which describes only phonon degrees of freedom including the electron-phonon coupling, where H_{ph} and H_{e-p} are given as follows:

$$H_{ph} = \sum_q \omega_q \left(b_q^\dagger b_q + \frac{1}{2} \right), \quad (A.1)$$

$$H_{e-p} = \int dx \left\{ gu(x) e^{-iQx} \psi_+^\dagger(x) \psi_-(x) + \text{h.c.} \right\}. \quad (A.2)$$

The lattice displacement $u(x)$ is expressed in terms of the phonon operators as

$$u(x) = \frac{1}{\sqrt{L}} \sum_q \frac{1}{\sqrt{2\rho_{ph}\omega_q}} \left(b_q + b_{-q}^\dagger \right) e^{-iqx}, \quad (A.3)$$

where L is the system length and ρ_{ph} is the mass density. With the Fourier transform of $\psi_\pm(x)$:

$$\psi_\pm(k) = \frac{1}{\sqrt{L}} \int dx e^{-ikx} \psi_\pm(x), \quad (A.4)$$

it is convenient to rewrite H_{e-p} as

$$H_{e-p} = \frac{1}{\sqrt{L}} \sum_{k,q} \frac{g}{\sqrt{2\rho_{ph}\omega_{Q+q}}} \left(b_{Q+q} + b_{-(Q+q)}^\dagger \right) \times \left\{ \psi_+^\dagger(k_+) \psi_-(k_-) + \psi_-^\dagger(k_-) \psi_+(k_+) \right\}, \quad (A.5)$$

where $k_\pm = k \pm q/2$. From the Heisenberg equation for the phonon operators, we can show that

$$\begin{aligned} & (\omega_Q^2 + \partial_t^2) \left\langle b_{Q+q}(t) + b_{-(Q+q)}^\dagger(t) \right\rangle \\ &= \frac{1}{\sqrt{L}} \frac{g(-2\omega_Q)}{\sqrt{2\rho_{ph}\omega_Q}} \sum_k \left\langle \psi_-^\dagger(k_-, t) \psi_+(k_+, t) \right\rangle. \end{aligned} \quad (A.6)$$

Performing the inverse Fourier transformation of the above equation and using the relation $g\langle u \rangle e^{-iQx} = \Delta - \zeta$, we obtain

$$\begin{aligned} & (1 + \omega_Q^{-2} \partial_t^2) [\Delta(x, t) - \zeta(x)] \\ &= i \frac{g^2}{2\rho_{ph}\omega_Q^2} G_{+-}^K(x, t; x', t') \Big|_{x' \rightarrow x, t' \rightarrow t}. \end{aligned} \quad (A.7)$$

Ignoring the irrelevant term with ∂_t^2 and rewriting the prefactor in the right-hand side in terms of the dimensionless coupling constant

$$\lambda = \frac{g^2}{8\pi\rho_{ph}\omega_Q^2 v_F}, \quad (A.8)$$

we finally arrive at the self-consistency equation [Eq. (19)].

Appendix B: Decomposition of Eq. (25)

We decompose Eq. (25) to yield a set of four equations for the matrix elements of \hat{g}^R . The resulting equations are as follows:

$$\begin{aligned} & i(\partial_t + v_F \partial_x) g^R + \Delta^* f^R + \Delta \bar{f}^R + i(ev_F \dot{A} + \dot{\Phi}) \partial_\epsilon g^R \\ & + \frac{i}{2} \left(\dot{\Delta}^* \partial_\epsilon f^R - \dot{\Delta} \partial_\epsilon \bar{f}^R \right) - \frac{1}{8} \left(\ddot{\Delta}^* \partial_\epsilon^2 f^R + \ddot{\Delta} \partial_\epsilon^2 \bar{f}^R \right) = 0, \end{aligned} \quad (B.1)$$

$$i(-\partial_t + v_F \partial_x) \bar{g}^R - \Delta^* f^R - \Delta \bar{f}^R + i(ev_F \dot{A} - \dot{\Phi}) \partial_\epsilon \bar{g}^R$$

$$+ \frac{i}{2} \left(\dot{\Delta}^* \partial_\epsilon f^R - \dot{\Delta} \partial_\epsilon \bar{f}^R \right) + \frac{1}{8} \left(\ddot{\Delta}^* \partial_\epsilon^2 f^R + \ddot{\Delta} \partial_\epsilon^2 \bar{f}^R \right) = 0, \quad (\text{B-2})$$

$$2(\epsilon - \Phi) f^R + i v_F \partial_x f^R - \Delta(g^R - \bar{g}^R) + i e v_F \dot{A} \partial_\epsilon f^R - \frac{i}{2} \dot{\Delta} (\partial_\epsilon g^R + \partial_\epsilon \bar{g}^R) + \frac{1}{8} \ddot{\Delta} (\partial_\epsilon^2 g^R - \partial_\epsilon^2 \bar{g}^R) + \frac{i}{2\tau} (g^R - \bar{g}^R) f^R = 0, \quad (\text{B-3})$$

$$- 2(\epsilon - \Phi) \bar{f}^R + i v_F \partial_x \bar{f}^R - \Delta^*(g^R - \bar{g}^R) + i e v_F \dot{A} \partial_\epsilon \bar{f}^R + \frac{i}{2} \dot{\Delta}^* (\partial_\epsilon g^R + \partial_\epsilon \bar{g}^R) + \frac{1}{8} \ddot{\Delta}^* (\partial_\epsilon^2 g^R - \partial_\epsilon^2 \bar{g}^R) - \frac{i}{2\tau} (g^R - \bar{g}^R) \bar{f}^R = 0. \quad (\text{B-4})$$

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